

# $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$ decays in Resonance Chiral Theory

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## Abstract

The hadronization structure of  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  decays is analyzed within the framework of Resonance Chiral Theory, considering only the contribution of the lightest meson resonances at leading order in the  $1/N_C$  expansion. After imposing the asymptotic behavior of vector spectral functions ruled by QCD, unknown effective couplings are determined by fitting the  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  branching ratio and decay spectrum to recent data. Predictions for the partner decay  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$  and the low-energy behavior of the cross section  $\sigma(e^+ e^- \rightarrow \eta \pi^+ \pi^-)$  are also discussed.

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# 1 Introduction

Tau decays represent an ideal benchmark where to analyze diverse topics in elementary particle physics [1]. In particular, semileptonic decay channels  $\tau^- \rightarrow H^- \nu_\tau$ , where  $H$  is some hadronic state, allow a rather clean theoretical analysis of the hadronization of the  $V - A$  currents in presence of QCD interactions, since there is no hadron pollution to the leptonic current. Thus, these processes provide a suitable tool to find out intrinsic properties of the involved hadron resonances [2, 3, 4, 5, 6]. In this work we concentrate on the analysis of  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decays. For these channels the contributions of scalar and pseudoscalar resonances are expected to be negligible, since they turn out to be forbidden at tree level by symmetry arguments, such as  $G$ -parity conservation. In the limit of isospin symmetry the corresponding amplitudes are driven by the vector current, allowing a precise study of the couplings in the odd-intrinsic parity sector.

Concerning the theoretical description, it is well known that in the very low-energy domain ( $E \ll M_\rho$ , where  $M_\rho$  is the mass of the  $\rho(770)$  meson) Chiral Perturbation Theory ( $\chi$ PT) [7] is the adequate tool to describe hadronic  $\tau$  decays [8]. However, this approach fails when the invariant mass of the hadronic state becomes comparable with the mass of the lightest vector and/or axial-vector resonances, therefore a new strategy is needed in order to enlarge the domain of applicability of  $\chi$ PT to higher energies. One way out in this sense is to abandon the Lagrangian approach: one can model  $\tau$  decay amplitudes by taking the lowest order (LO)  $\chi$ PT results to fix the normalization of the form factors at low energies, incorporating the dominant vector and axial-vector meson resonance exchanges by modulating the amplitudes with *ad hoc* Breit-Wigner functions [2, 3, 4, 5, 9]. However, it can be seen that in the low-energy limit this approach is in general not consistent with next-to-leading order (NLO)  $\chi$ PT [7], hence the usage of this procedure to reproduce QCD-ruled amplitudes is questionable [10, 11]. An alternative approach is to include the lightest resonances as active degrees of freedom in the theory. This is done in the so-called Resonance Chiral Theory (R $\chi$ T) [12, 13, 14, 15], where the inclusion of resonances is carried out together with an expansion in the inverse of the number of colors ( $N_C$ ) [16, 17, 18, 19]. In fact, at the lowest order in the  $1/N_C$  expansion one gets from QCD an effective theory that includes a spectrum of infinite zero-width states. The goal of R $\chi$ T is to go beyond this limit, incorporating—in a way consistent with QCD symmetry requirements—a few finite-width resonant states that dominate the processes under study.

A basic assumption in R $\chi$ T is that the lightest resonant states are the dominant ones in low-energy phenomenology. In this way, for a given process it should be sufficient to introduce only the lightest resonance multiplet carrying the appropriate quantum numbers, while the inclusion of higher states can be carried out as a correction [20, 21]. On the other hand, the theory is built upon some fundamental QCD-based features: the effective interactions have to satisfy QCD symmetries, the low-energy behavior has to be consistent with  $\chi$ PT, and the asymptotic behavior of Green functions and associated form factors has to satisfy QCD constraints. These requirements imply several relations among the effective couplings that render the theory predictive. The aim of this work is to study within this framework the decays  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$ ,  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$ , and the low-energy limit of the cross section  $\sigma(e^+ e^- \rightarrow \eta \pi^+ \pi^-)$ .

The article is organized as follows: In Sect. 2 we recall the fundamentals of Resonance Chiral Theory (see e.g. Ref. [22]) to explain our methodology. The relevant hadronic form factors for the decays under study are given in Sect. 3. In Sect. 4 we derive the QCD-ruled high-energy constraints on the couplings, which reduce the number of unknowns to only four. In Sect. 5 we show that two of these unknowns can be bounded from other phenomenological studies performed within the same framework. In this way we end up with two unknown couplings, which appear to be highly correlated [23, 24]. The possible values of these couplings are analyzed by fitting experimental data on the differential decay distribution of  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  and taking into account the present upper limit on the branching ratio for  $\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau$ . The low-energy behavior of the cross section  $\sigma(e^+e^- \rightarrow \eta\pi^+\pi^-)$  is also discussed. Our conclusions are presented in Sect. 6. Finally, in Appendices A and B we analyze other possible contributions to the decay amplitudes and quote some useful isospin relations.

## 2 Theoretical framework

The effective Lagrangian in  $R\chi T$  is basically ruled by the approximate chiral symmetry of light-flavored QCD—which drives the interaction of light pseudoscalar mesons—and the  $SU(3)_V$  assignments of resonance multiplets [12, 14]. As we will see, for the processes under consideration it is possible to achieve a good agreement with present experimental data without the inclusion of excited multiplets. Moreover, it is seen that vector meson dominance (VMD) turns out to be a good approximation [12], since spin-zero resonance contributions vanish at tree level in the very accurate isospin symmetry limit (see App. A). In the case of  $\tau$  decays, owing to the relatively large  $\tau$  mass it occurs that several resonances reach their on-shell condition when the amplitudes are integrated over the full phase space. The corresponding pole singularities can be regularized by including finite (energy-dependent) resonance widths, thus departing from the lowest order in the  $1/N_C$  expansion. Here we adopt the prescription in Ref. [25], where energy-dependent resonance widths have been calculated in a well-defined way within  $R\chi T$ .

We will work out  $\tau^- \rightarrow \eta^{(\prime)}\pi^-\pi^0\nu_\tau$  decays considering exact isospin symmetry. In this limit the processes are driven only by the vector current (see Sect. 3), and appear to be dominated by the contributions of the  $\rho(770)$  resonance. The relevant effective Lagrangian in  $R\chi T$  reads :

$$\begin{aligned} \mathcal{L}_{R\chi T} \doteq & \mathcal{L}_{WZW} + \mathcal{L}_{\text{kin}}^V + \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle \\ & + i \frac{G_V}{\sqrt{2}} \langle V_{\mu\nu} u^\mu u^\nu \rangle + \sum_{i=1}^7 \frac{c_i}{M_V} \mathcal{O}_{VJP}^i + \sum_{i=1}^4 d_i \mathcal{O}_{VVP}^i + \sum_{i=1}^5 \frac{g_i}{M_V} \mathcal{O}_{VPPP}^i, \end{aligned} \quad (1)$$

where all coupling constants are real,  $F$  and  $M_V$  being the pion decay constant and the mass of the lightest vector meson resonances, respectively. We follow here the notation in Refs. [11, 12, 26]<sup>1</sup>. Accordingly,  $\langle \rangle$  stands for trace in flavor

<sup>1</sup>In Ref. [24], two additional operators ( $\tilde{\mathcal{O}}_{VJP}^8$  and  $\tilde{\mathcal{O}}_{VVP}^5$ ) have been found when the singlet  $\langle VVP \rangle$  Green function is considered in addition to the octet one in the  $p^2 \sim m_q \sim 1/N_C$  counting. In App. A we show that they do not contribute to the hadronic tau decays studied here.

space, and  $u^\mu$ ,  $\chi_+$  and  $f_+^{\mu\nu}$  are defined by

$$\begin{aligned} u^\mu &= i u^\dagger D^\mu U u^\dagger, \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \\ f_\pm^{\mu\nu} &= u^\dagger F_{L,R}^{\mu\nu} u^\dagger \pm u F_R^{\mu\nu} u, \end{aligned} \quad (2)$$

where  $u$  ( $U = u^2$ ),  $\chi$  and  $F_{L,R}^{\mu\nu}$  are  $3 \times 3$  matrices that contain light pseudoscalar fields, current quark masses and external left and right currents, respectively. The matrix  $V^{\mu\nu}$  includes the lightest vector meson multiplet, and  $\mathcal{L}_{\text{kin}}^V$  stands for the resonance kinetic term. The first term in Eq. (1) is the Wess-Zumino-Witten interaction Lagrangian [27, 28], which governs the decay amplitudes studied here in the limit of low hadron momenta. The part of this interaction that contributes to the processes considered here reads

$$\begin{aligned} \mathcal{L}_{WZW} &\doteq -\frac{iN_C}{48\pi^2} \epsilon_{\mu\nu\alpha\beta} \langle \Sigma_L^\mu U^\dagger \partial^\nu r^\alpha U l^\beta + \Sigma_L^\mu l^\nu \partial^\alpha l^\beta \\ &\quad + \Sigma_L^\mu \partial^\nu l^\alpha l^\beta - (L \leftrightarrow R) \rangle, \end{aligned} \quad (3)$$

where  $\Sigma_{L,R}$  are given by  $\Sigma_L^\mu = U^\dagger \partial^\mu U$ ,  $\Sigma_R^\mu = U \partial^\mu U^\dagger$ , and  $l^\alpha$  and  $r^\alpha$  are left and right external currents. Finally, the operators  $\mathcal{O}_{VJP}^i$ ,  $\mathcal{O}_{VVP}^i$  and  $\mathcal{O}_{VPP}^i$  in Eq. (1) are given by :

*VJP* terms

$$\begin{aligned} \mathcal{O}_{VJP}^1 &= \epsilon_{\mu\nu\rho\sigma} \langle \{V^{\mu\nu}, f_+^{\rho\alpha}\} \nabla_\alpha u^\sigma \rangle, \\ \mathcal{O}_{VJP}^2 &= \epsilon_{\mu\nu\rho\sigma} \langle \{V^{\mu\alpha}, f_+^{\rho\sigma}\} \nabla_\alpha u^\nu \rangle, \\ \mathcal{O}_{VJP}^3 &= i \epsilon_{\mu\nu\rho\sigma} \langle \{V^{\mu\nu}, f_+^{\rho\sigma}\} \chi_- \rangle, \\ \mathcal{O}_{VJP}^4 &= i \epsilon_{\mu\nu\rho\sigma} \langle V^{\mu\nu} [f_-^{\rho\sigma}, \chi_+] \rangle, \\ \mathcal{O}_{VJP}^5 &= \epsilon_{\mu\nu\rho\sigma} \langle \{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\} u^\sigma \rangle, \\ \mathcal{O}_{VJP}^6 &= \epsilon_{\mu\nu\rho\sigma} \langle \{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\} u^\nu \rangle, \\ \mathcal{O}_{VJP}^7 &= \epsilon_{\mu\nu\rho\sigma} \langle \{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\} u_\alpha \rangle; \end{aligned} \quad (4)$$

*VVP* terms

$$\begin{aligned} \mathcal{O}_{VVP}^1 &= \epsilon_{\mu\nu\rho\sigma} \langle \{V^{\mu\nu}, V^{\rho\alpha}\} \nabla_\alpha u^\sigma \rangle, \\ \mathcal{O}_{VVP}^2 &= i \epsilon_{\mu\nu\rho\sigma} \langle \{V^{\mu\nu}, V^{\rho\sigma}\} \chi_- \rangle, \\ \mathcal{O}_{VVP}^3 &= \epsilon_{\mu\nu\rho\sigma} \langle \{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\} u^\sigma \rangle, \\ \mathcal{O}_{VVP}^4 &= \epsilon_{\mu\nu\rho\sigma} \langle \{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\} u_\alpha \rangle; \end{aligned} \quad (5)$$

$VPPP$  terms

$$\begin{aligned}
\mathcal{O}_{VPPP}^1 &= i \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} (h^{\alpha\gamma} u_\gamma u^\beta - u^\beta u_\gamma h^{\alpha\gamma}) \rangle, \\
\mathcal{O}_{VPPP}^2 &= i \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} (h^{\alpha\gamma} u^\beta u_\gamma - u_\gamma u^\beta h^{\alpha\gamma}) \rangle, \\
\mathcal{O}_{VPPP}^3 &= i \varepsilon_{\mu\nu\alpha\beta} \langle V^{\mu\nu} (u_\gamma h^{\alpha\gamma} u^\beta - u^\beta h^{\alpha\gamma} u_\gamma) \rangle, \\
\mathcal{O}_{VPPP}^4 &= \varepsilon_{\mu\nu\alpha\beta} \langle \{ V^{\mu\nu}, u^\alpha u^\beta \} \chi_- \rangle, \\
\mathcal{O}_{VPPP}^5 &= \varepsilon_{\mu\nu\alpha\beta} \langle u^\alpha V^{\mu\nu} u^\beta \chi_- \rangle,
\end{aligned} \tag{6}$$

where  $h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$ . The covariant derivative  $\nabla_\mu$  involves pseudoscalar meson fields and  $l^\alpha, r^\alpha$  external currents. Its explicit expression can be found in Ref. [12].

The nonet of vector resonances  $V$  is described here using the antisymmetric tensor formulation. In the context of VMD [14], this is shown to be consistent with the usage of the  $\chi$ PT Lagrangian for light pseudoscalar mesons up to  $\mathcal{O}(p^2)$  in the even-intrinsic parity sector and up to  $\mathcal{O}(p^4)$  in the odd-intrinsic parity sector [15].

### 3 Form factors in $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$

In the Standard Model,  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decay amplitudes can be written as

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}} V_{ud} \bar{u}_{\nu_\tau} \gamma^\mu (1 - \gamma_5) u_\tau \mathcal{H}_\mu, \tag{7}$$

where  $V_{ud} \simeq \cos \theta_C$  is the relevant Cabibbo-Kobayashi-Maskawa mixing and  $\mathcal{H}_\mu$  is the hadron matrix element of the left-handed QCD current  $(V - A)_\mu$ . In general, for a decay of a  $\tau$  lepton into three pseudoscalar mesons the hadronic tensor  $\mathcal{H}_\mu$  can be written as [6]

$$\begin{aligned}
\langle h_1(p_1) h_2(p_2) h_3(p_3) | (V - A)^\mu | 0 \rangle &= F_1^A(Q^2, s_1, s_2) V_1^\mu \\
&+ F_2^A(Q^2, s_1, s_2) V_2^\mu + i F_3^V(Q^2, s_1, s_2) V_3^\mu + F_4^A(Q^2, s_1, s_2) Q^\mu,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
V_1^\mu &= \left( g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_1 - p_3)_\nu, \quad V_2^\mu = \left( g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) (p_2 - p_3)_\nu, \\
V_3^\mu &= \varepsilon^{\mu\alpha\beta\gamma} p_{1\alpha} p_{2\beta} p_{3\gamma}, \quad Q^\mu = (p_1 + p_2 + p_3)^\mu, \quad s_i = (Q - p_i)^2.
\end{aligned} \tag{9}$$

The upper indices in the form factors indicate the participating currents, either the axial-vector ( $A$ ), or the vector one ( $V$ ). The form factors  $F_1^A$  and  $F_2^A$  drive a transition to hadronic states with quantum numbers  $J^P = 1^+$ , while  $F_3^V$  and  $F_4^A$  correspond to outgoing states with  $J^P = 1^-$  and  $J^P = 0^-$ , respectively. Let us focus on the amplitude for the transition  $\tau^- \rightarrow \eta_8(p_1) \pi^-(p_2) \pi^0(p_3) \nu_\tau$ , considering the limit of exact isospin symmetry. First of all, it is easy to see that for this process the axial-vector form factors vanish from  $G$ -parity conservation, therefore the dynamics will be essentially determined by the form factor  $F_3^V$ .

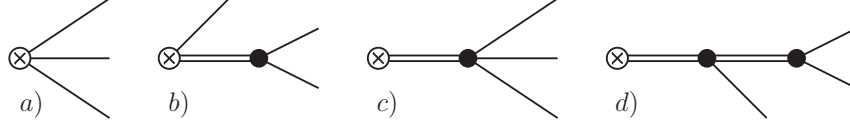


Figure 1: Topologies contributing to the final hadron state in  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decays in the  $N_C \rightarrow \infty$  limit. Crossed circles indicate QCD vector current insertions. Single lines represent pseudoscalar mesons ( $\eta$ ,  $\pi$ ) while double lines stand for  $\rho$ -resonance intermediate states.

From the effective Lagrangian in Eq. (1), the diagrams that contribute to  $F_3^V$  are those represented in Fig. 1, where single solid lines correspond to  $\pi$  and  $\eta$  mesons and double lines to the  $\rho(770)$  resonance. The corresponding contributions to the vector form factor read

$$F_3^{V(a)}{}_{(\eta_8 \pi \pi)} = \frac{N_C}{6 \sqrt{6} \pi^2 F^3}, \quad (10)$$

$$F_3^{V(b)}{}_{(\eta_8 \pi \pi)} = \frac{8 G_V}{\sqrt{3} F^3 M_V} \frac{1}{M_\rho^2 - s_1} [c_{125} Q^2 - c_{1256} s_1 + c_{1235} m_\eta^2 + 8 c_3 (m_\pi^2 - m_\eta^2)], \quad (11)$$

$$F_3^{V(c)}{}_{(\eta_8 \pi \pi)} = -\frac{16 F_V}{\sqrt{3} M_V F^3} \frac{1}{M_\rho^2 - Q^2} [g_{123} s_1 - g_2 (Q^2 + 2 m_\pi^2 - m_\eta^2) - (g_1 - g_3) 2 m_\pi^2 + g_{45} m_\pi^2], \quad (12)$$

$$F_3^{V(d)}{}_{(\eta_8 \pi \pi)} = -\frac{8 \sqrt{2} F_V G_V}{\sqrt{3} F^3} \frac{1}{M_\rho^2 - Q^2} \frac{1}{M_\rho^2 - s_1} \times [d_3 (Q^2 + s_1) + (d_{12} - d_3) m_\eta^2 + 8 d_2 (m_\pi^2 - m_\eta^2)], \quad (13)$$

where we have defined

$$\begin{aligned} c_{125} &= c_1 - c_2 + c_5, \\ c_{1256} &= c_1 - c_2 - c_5 + 2 c_6, \\ c_{1235} &= c_1 + c_2 + 8 c_3 - c_5, \\ g_{123} &= g_1 + 2 g_2 - g_3, \\ g_{45} &= 2 g_4 + g_5, \\ d_{12} &= d_1 + 8 d_2. \end{aligned} \quad (14)$$

The amplitude for the  $\tau$  decay into the  $\eta_0 \pi^- \pi^0$  hadronic state can be read from Eqs. (10) to (13) by simply multiplying  $F_3^{V(a,b,c,d)}{}_{(\eta_8 \pi \pi)}$  by  $\sqrt{2}$ . Then, the matrix elements for the decays into the physical hadronic states  $\eta \pi^- \pi^0$  and  $\eta' \pi^- \pi^0$  can be obtained by considering  $\eta_8 - \eta_0$  mixing. Here we will consider a double angle mixing scheme [29], which is consistent with the large- $N_C$  expansion [30]. Using a notation similar to that in Ref. [31], the SU(3) octet and singlet fields are collected in a doublet  $\eta_B^T \equiv (\eta_8, \eta_0)$ , while the physical fields are included in  $\eta_P^T \equiv (\eta, \eta')$ . These doublets are related by the transformation

$\eta_B = (\mathcal{M})^T \eta_P$ , where [31]

$$\mathcal{M} = \begin{pmatrix} \cos \theta_P (1 - \delta_8/2) + \sin \theta_P \delta_{80}/2 & -\sin \theta_P (1 - \delta_0/2) - \cos \theta_P \delta_{80}/2 \\ \sin \theta_P (1 - \delta_8/2) - \cos \theta_P \delta_{80}/2 & \cos \theta_P (1 - \delta_0/2) - \sin \theta_P \delta_{80}/2 \end{pmatrix}. \quad (15)$$

In the framework of R $\chi$ T, the parameters  $\delta_8$ ,  $\delta_0$  and  $\delta_{80}$  can be in fact derived from an effective Lagrangian that involves scalar resonances [12]. If the latter are organized in a U(3) matrix  $S$ , from the lowest order Lagrangian

$$\mathcal{L}^S = c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle \quad (16)$$

one gets

$$\delta_8 = \frac{8c_d c_m}{M_S^2} \frac{M_8^2}{F^2}, \quad \delta_0 = \frac{8c_d c_m}{M_S^2} \frac{M_0^2}{F^2}, \quad \delta_{80} = \frac{8c_d c_m}{M_S^2} \frac{M_{80}^2}{F^2}, \quad (17)$$

where<sup>2</sup>

$$\begin{aligned} M_8^2 &= \frac{1}{3} (4M_K^2 - M_\pi^2), \\ M_0^2 &= \frac{1}{3} (2M_K^2 + M_\pi^2), \\ M_{80}^2 &= -\frac{2\sqrt{2}}{3} (M_K^2 - M_\pi^2). \end{aligned} \quad (18)$$

Here we take for  $M_\pi$  and  $M_K$  the isospin averaged values of the pion and kaon masses, neglecting higher order corrections in the combined chiral and  $1/N_C$  expansion. In addition we assume  $c_d c_m = F^2/4$  [33, 34], which is required by high-energy QCD in the  $N_C \rightarrow \infty$  limit. Finally, from  $\eta - \eta'$  phenomenology we take  $M_S \simeq 0.980$  GeV and  $\theta_P = (-13.3 \pm 0.5)^\circ$  [35].

Given the form factors,  $F_3^V(Q^2, s_1, s_2)$ , the spectral functions for the decays  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  are finally given by

$$\begin{aligned} \frac{d\Gamma}{dQ^2} &= \frac{G_F^2 |V_{ud}|^2}{128(2\pi)^5 M_\tau} \left( \frac{M_\tau^2}{Q^2} - 1 \right)^2 \frac{1}{3} \left( 1 + 2 \frac{Q^2}{M_\tau^2} \right) \\ &\times \int_{(m_\eta + m_\pi)^2}^{(\sqrt{Q^2} - m_\pi)^2} ds_2 \int_{t_-(Q^2, s_2)}^{t_+(Q^2, s_2)} ds_1 W_B(Q^2, s_1, s_2), \end{aligned} \quad (19)$$

where the relevant structure function  $W_B$  [6] is defined by  $W_B(Q^2, s_1, s_2) = V_3^2 |F_3^V(Q^2, s_1, s_2)|^2$  and the limits of the integral over  $s_1$  are

$$\begin{aligned} t_\pm(Q^2, s_2) &= \frac{1}{4s_2} \left\{ (Q^2 + m_\eta^2 - 2m_\pi^2)^2 \right. \\ &\quad \left. - \left[ \lambda^{1/2}(Q^2, s_2, m_\pi^2) \mp \lambda^{1/2}(m_\eta^2, m_\pi^2, s_2) \right]^2 \right\}, \end{aligned} \quad (20)$$

with  $\lambda(a, b, c) = (a + b - c)^2 - 4ab$ . We have neglected here the neutrino mass.

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<sup>2</sup>The fully dominant contribution to the  $\eta'$  mass is not due to current quark masses but to the  $U(1)_A$  anomaly [32], through the topological susceptibility of gluodynamics. Hence we keep  $\theta_P$  as a free parameter, to be fitted from phenomenology.

## 4 Short distance constraints on the couplings

The above form factors depend on several combinations of coupling constants, besides the  $\rho$  mass and the pion decay constant. The values of these parameters are not provided by the effective theory, and their determination from the underlying QCD theory is still an open problem. However, one can get information on the effective couplings by assuming that the resonance region provides a bridge between the chiral and perturbative regimes, even when one does not include the full resonance spectrum [14]. This is implemented by matching the high-energy behavior of Green functions (or related form factors) evaluated within the resonance theory with asymptotic results obtained in perturbative QCD [11, 14, 15, 21, 24, 26, 36]. In particular, it has been shown that the analysis of the two-point Green functions  $\Pi_{A,V}$  [14] and the three-point Green function  $VVP$  of QCD currents (with the inclusion of only one multiplet of vector resonances) [26] leads to the following constraints in the  $N_C \rightarrow \infty$  limit:

- i) By demanding that the two-pion vector form factor vanishes at high momentum transfer one obtains the condition  $F_V G_V = F^2$  [14].
- ii) The analysis of the  $VVP$  Green function [26] leads to the following results for the couplings in Eqs. (11), (12) and (13) :

$$\begin{aligned}
c_{125} &= 0, \\
c_{1235} &= 0, \\
c_{1256} &= -\frac{N_C}{32\pi^2} \frac{M_V}{\sqrt{2}F_V}, \\
d_{12} &= -\frac{N_C}{64\pi^2} \frac{M_V^2}{F_V^2} + \frac{F^2}{4F_V^2}, \\
d_3 &= -\frac{N_C}{64\pi^2} \frac{M_V^2}{F_V^2} + \frac{F^2}{8F_V^2}.
\end{aligned} \tag{21}$$

On the other hand, it is possible to find additional constraints by requiring that the contributions of any intermediate hadronic state to the spectral function  $\text{Im}\Pi_V(Q^2)$  vanish in the limit  $Q^2 \rightarrow \infty$ . This is a reasonable assumption, since from perturbative QCD  $\text{Im}\Pi_V(Q^2)$  has to go to a constant value for  $Q^2 \rightarrow \infty$  [37], and the imaginary part of the two-point Green function can be understood as the sum of infinite intermediate hadronic states. Considering the intermediate  $\eta^{(\prime)}\pi\pi$  hadronic states one gets the following constraints on the coupling constants:

$$\begin{aligned}
c_{125} &= 0, \\
c_{1256} &= -\frac{N_C}{96\pi^2} \frac{M_V F_V}{\sqrt{2}F^2}, \\
d_3 &= -\frac{N_C}{192\pi^2} \frac{M_V^2}{F^2}, \\
g_{123} &= 0, \\
g_2 &= \frac{N_C}{192\pi^2} \frac{M_V}{\sqrt{2}F_V}.
\end{aligned} \tag{22}$$

It is worth to notice that relations (22) are in agreement with those found in a similar analysis carried out for  $\tau$  decays into  $2K\pi\nu_\tau$  [38, 39] and  $P^-\gamma\nu_\tau$  ( $P =$



$\pi, K$ ) states [40]. Comparing with Eqs. (21), we agree in the vanishing of  $c_{125}$ , while the constraints for  $c_{1256}$  and  $d_3$  cannot be simultaneously satisfied keeping agreement with their values in Eq. (22). Moreover, as stated in Ref. [38], it is seen that the expected vanishing of the  $\pi\gamma^*\gamma$  form factor at high- $q^2$  is obtained from Eqs. (22) but not from Eqs. (21). In any case, numerically the differences are small, and the impact of these couplings on the observables is rather mild<sup>3</sup>. Thus we choose to stick to our set of relations (22), using Eqs. (21) to fix the combinations  $c_{1235}$  and  $d_{12}$ , not obtained within our study. In this way, the analysis of short distance constraints allows to reduce significantly the number of unknown coupling constants in the form factors  $F_3^{V(\alpha)}$  quoted in Eqs. (10-13). To calculate the decay amplitudes for the processes  $\tau^- \rightarrow \eta^{(\prime)}\pi^-\pi^0\nu_\tau$  we end up with just four unknown parameters, namely  $F_V$ ,  $c_3$ ,  $g_{45}$  and  $d_2$ . As in the above mentioned analysis, we will take  $M_V = M_\rho$ .

## 5 Phenomenological analysis

In order to carry out a phenomenological analysis of  $\tau^- \rightarrow \eta^{(\prime)}\pi^-\pi^0\nu_\tau$  decays we take into account the available experimental information. In the case of  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  this includes the measured branching fraction  $\text{BR}(\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau) = (1.39 \pm 0.10) \times 10^{-3}$  [41], as well as the data on the corresponding spectral function obtained by Belle [42]. The process  $\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau$  has not been observed yet, hence we consider only the upper bound given by the PDG [41], namely  $\text{BR}(\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau) < 8.0 \times 10^{-5}$  at 90% confidence level.

As stated, the number of unknown parameters entering the vector form factor  $F_3^V$  in  $\text{R}\chi\text{T}$  can be reduced to four by means of the short-distance constraints obtained in Sect. 4. In addition, the values of  $F_V$  and  $g_{45}$  can be estimated within  $\text{R}\chi\text{T}$  from the phenomenological analysis of  $\tau^- \rightarrow (\pi\pi\pi)^-\nu_\tau$  and  $\omega \rightarrow \pi^+\pi^-\pi^0$ , respectively: the best fit to the  $\tau^- \rightarrow (\pi\pi\pi)^-\nu_\tau$  spectral function measured by ALEPH [43] corresponds to  $F_V = 0.180$  GeV [21] with an estimated error of  $\sim 15\%$ <sup>4</sup>, while from the  $\omega \rightarrow \pi^+\pi^-\pi^0$  branching ratio one gets  $g_{45} = -0.60 \pm 0.02$  [38]. In this way we are left with only two unknowns, namely the coupling constants  $c_3$  and  $d_2$ . Our goal is to be able to describe the available experimental information just by fitting these two parameters.

In Figure 2 we show the  $c_3 - d_2$  parameter region compatible with the PDG branching ratio for the mode  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  at the level of one sigma. A large correlation between both couplings can be appreciated, in agreement with Refs. [23, 24]<sup>5</sup>. Then, taking into account this allowed region for  $c_3$  and  $d_2$ , we have carried out a fit to Belle data [42] for the  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  spectral

<sup>3</sup>The introduction of additional resonances has a different effect on the short-distance relations obtained from the  $VVP$  Green function and from the imaginary part of the vector-vector correlator. While all new contributions to the correlator are positive definite, this is not true for the  $VVP$  Green function, where cancellations are allowed. Thus the outcome of both procedures may be different when the spectrum is restricted to the lowest-lying resonances. A convergence of both results should be recovered if the full tower of excited resonances is taken into account.

<sup>4</sup>Some theoretical analyses lead to the value  $F_V = \sqrt{3}F \sim 0.160$  GeV [40, 44, 45, 46, 47, 48, 49]. We have checked that a change of  $F_V$  within the range  $[0.160, 0.180]$  GeV does not affect significantly the results presented in this section.

<sup>5</sup>In particular, as noticed in Ref. [24], there is an anticorrelation between  $c_3 - d_2$  in  $\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau$  and in associated radiative decays. Therefore, the combined study could improve the determination of these couplings.

function. We find two  $\chi^2$  minima, located at  $(c_3, d_2) = (-0.018, 0.45)$  and  $(c_3, d_2) = (0.035, -0.70)$ , with  $\chi^2/\text{dof} = 3.80$  and  $4.34$ , respectively, where the statistical error is about 10%. These values are indicated in Fig. 2, where black strips correspond to the  $c_3 - d_2$  regions that keep  $\chi^2/\text{dof}$  within one unit far from the minima. The corresponding theoretical curves for the spectral function, together with experimental data, are shown in Fig. 3. We have also carried out a fit to the normalized spectral function, obtaining that the preferred values for  $(c_3, d_2)$  remain almost unchanged, while  $\chi^2/\text{dof}$  values get reduced to 3.0 for both minima. Finally, in order to account for the theoretical error of the high-energy predictions for the couplings  $c_{1256}$ ,  $d_3$ ,  $g_2$  and  $d_{12}$ , we have also fitted the data allowing these coupling combinations to vary within  $\pm 1/3$  of the values obtained from Eqs. (21) and (22)<sup>6</sup>. Noteworthy, the  $\chi^2$  value does not get reduced, which can be taken as an indication that our short-distance relations (obtained at leading order in  $1/N_C$ ) lead to an appropriate effective Lagrangian to reproduce the experimental observations.

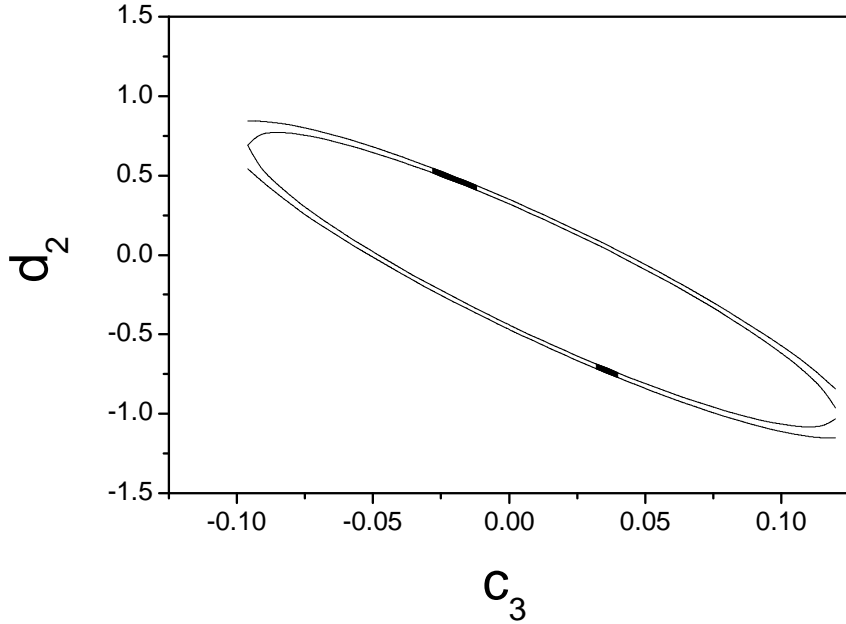


Figure 2: Contour in the  $c_3 - d_2$  plane compatible with the branching ratio of the decay  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  at the level of one standard deviation. The black strips highlight the two regions that yield lowest values of  $\chi^2/\text{dof}$  (within one unit) from a fit to Belle data [42] on the corresponding spectral function.

<sup>6</sup>We have also considered nonvanishing values for the coupling  $c_{1235}$ , which should be zero according to Eq. (21). Notwithstanding, have kept  $c_{125}$  and  $g_{123}$  equal to zero. Indeed, if  $c_{125} \neq 0$  the Brodsky-Lepage behavior [50] of the form factor is violated, and  $\text{Im}\Pi_V$  goes, asymptotically, as  $Q^6 \log(Q^2/M_V^2)$ ; if  $g_{123} \neq 0$ , the asymptotic growth goes as  $\mathcal{O}(Q^6)$ . Varying  $c_{1235}$  in the range  $[-0.05, 0.05]$  does not improve the fit.

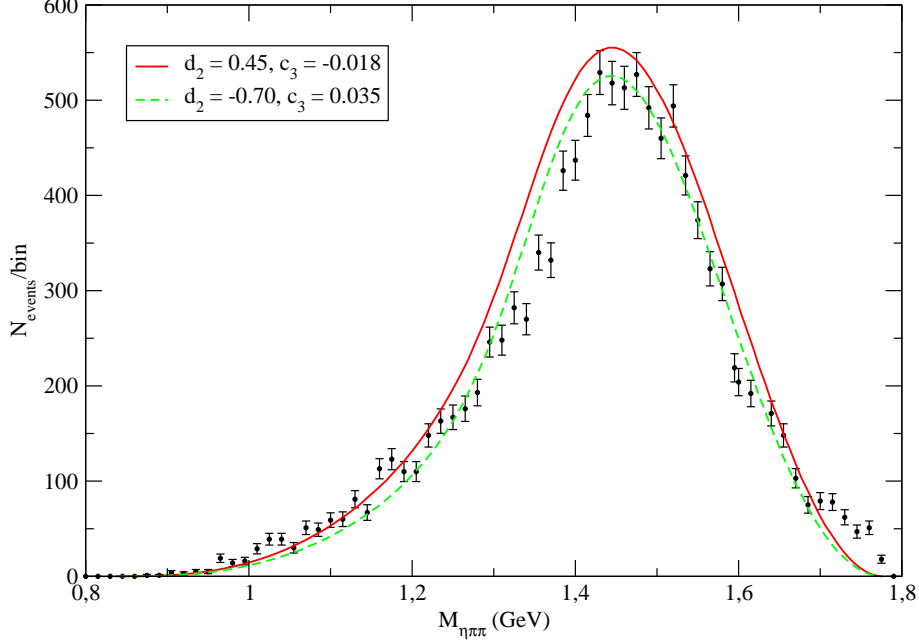


Figure 3: Theoretical curves fitting the spectral function for  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  decay, compared to experimental data [42].

Considering the fitted values for  $c_3$  and  $d_2$ , we can calculate the corresponding predictions for the  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$  branching ratio. The results are shown in Fig. 4, where we have taken  $c_3$  as the independent parameter. It is seen that the predictions are somewhat above the 90% confidence level upper bound quoted by the PDG, which is indicated by the shadowed region in the figure. However, the result corresponding to  $c_3 = -0.018$  turns out to be rather close to the upper bound; in fact, compatibility is achieved if the width of the  $c_3 - d_2$  band is enlarged considering two standard deviations in the measured value of  $\text{BR}(\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau)$ . Future, more precise measurements of the  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$  process should indicate whether our slight discrepancy arises from a weakness in the theoretical assumptions (e.g. treatment of  $\eta_8 - \eta_0$  mixing, effect of excited resonances, SU(3) breaking terms in the Lagrangian [51]) or it just reflects an issue in the detection of this  $\tau$  decay mode. In this regard, we emphasize the importance of making global fits with unified and consistent treatments of all hadronic tau decay channels from misunderstood backgrounds. The improvement in the most relevant hadronic matrix elements in TAUOLA [52, 53] may be a key tool in this sense.

Finally, our analysis can be used to predict  $\sigma(e^+e^- \rightarrow \eta \pi^+ \pi^-)$  in the low energy region (conversely, one could in general use data on  $e^+e^-$  annihilation into hadronic states to get predictions for the corresponding semileptonic tau

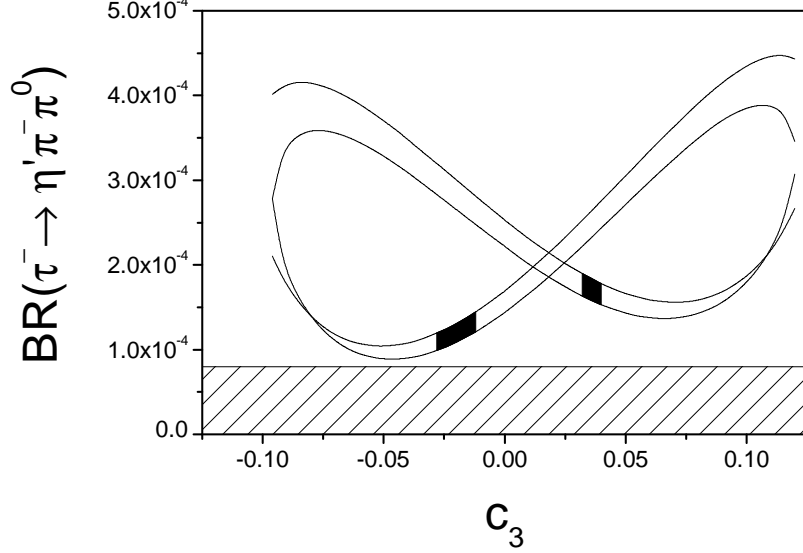


Figure 4: Prediction for the branching ratio of the decay  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$  consistent (within one sigma) with the  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  branching ratio quoted by the PDG. The horizontal line, corresponding to  $\text{BR}(\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau) = 0.8 \times 10^{-4}$ , represents the current PDG bound. The notation for the black strip is the same as in Fig. 2.

decays [54, 55]<sup>7</sup>). The relation between this cross section and the  $\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau$  spectral function is detailed in Appendix B. One gets

$$\frac{d\Gamma(\tau^- \rightarrow \eta \pi^- \pi^0 \nu_\tau)}{dQ^2} = 2 f(Q^2) \sigma(e^+ e^- \rightarrow \eta \pi^+ \pi^-), \quad (23)$$

where  $f(Q^2)$  is given by

$$f(Q^2) = \frac{G_F^2 |V_{ud}|^2}{384 (2\pi)^5 M_\tau} \left( \frac{M_\tau^2}{Q^2} - 1 \right)^2 \left( 1 + 2 \frac{Q^2}{M_\tau^2} \right) \left( \frac{\alpha^2}{48\pi} \right)^{-1} Q^6. \quad (24)$$

In Fig. 5 we quote our predictions for the cross section, in comparison with low-energy  $e^+ e^-$  data obtained in various experiments<sup>8</sup>. We notice that although the  $\eta'$  meson decays to  $\eta \pi^+ \pi^-$  with a fraction of about 45%, there is no significant contamination from the chain  $\sigma(e^+ e^- \rightarrow \eta' \gamma^* \rightarrow \eta \pi^+ \pi^-)$  since, due to  $C$  parity, this occurs at NLO in powers of the electromagnetic coupling  $\alpha$ . From the figure it is seen that our results are consistent with experimental data up to a center of mass energy of about 1.4 GeV. In fact, one should not expect

<sup>7</sup>A more elaborated dedicated approach, also based in R $\chi$ T, has been developed for  $\sigma(e^+ e^- \rightarrow \eta/\pi^0 \pi^+ \pi^-)$  [56].

<sup>8</sup>We note that in this neutral current process there are additional contributions from new operators  $\mathcal{O}_{\text{VJP}}^8$  and  $\mathcal{O}_{\text{VVP}}^5$ , see Ref. [24], which implies the introduction of two additional unknown couplings. However, these terms are suppressed in the large- $N_C$  limit in the standard counting (see discussion in App. A).

our treatment to be valid beyond this energy region, where effects of excited states should be sizeable and there is no phase space suppression as in  $\tau$  decay spectral functions.

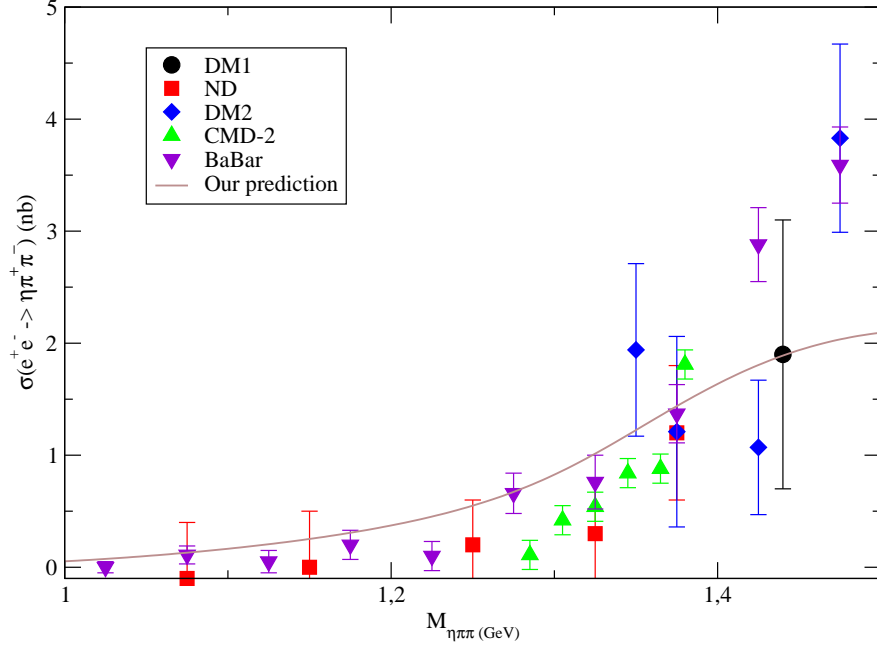


Figure 5: Prediction for  $\sigma(e^+e^- \rightarrow \eta\pi^+\pi^-)$  low-energy behavior from our analysis of  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  decays, in comparison with DM1 [57], ND [58], DM2 [59], CMD-2 [60] and BaBar [61] data.

## 6 Conclusions

We have worked out the decays  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  and  $\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau$  within the framework of Resonance Chiral Theory. The theoretical analysis has been based on the large- $N_C$  expansion of QCD, the low-energy limit given by  $\chi$ PT and the appropriate asymptotic behavior of the form factors, which helps to fix most of the initially unknown effective couplings. Indeed, after taking into account information acquired in the previous related studies,  $\tau^- \rightarrow \eta^{(\prime)}\pi^-\pi^0\nu_\tau$  amplitudes can be written in terms of only two unknown parameters.

We have carried out a phenomenological analysis taking into account the experimental data for the branching ratio and the spectrum of the decay  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$ , as well as the present upper bound for the  $\tau^- \rightarrow \eta'\pi^-\pi^0\nu_\tau$  branching fraction. A fit to the data allows to determine two preferred sets of values for the unknown parameters  $c_3$  and  $d_2$  in the effective Lagrangian, namely  $(c_3, d_2) = (-0.018, 0.45)$  and  $(0.035, -0.88)$ , which lead to a reasonable overall description of the spectrum. The former set seems to be favored by the predic-

tions for the branching ratio  $\text{BR}(\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau)$ , although in both cases the theoretical values appear to be somewhat above the present experimental upper bound. Finally, using isospin symmetry, these results can be used to get a prediction for the low-energy behavior of the cross section  $\sigma(e^+ e^- \rightarrow \eta \pi^+ \pi^-)$ . The results are in good agreement with the available experimental information, and the approach can be useful [56] for the implementation of the related hadronic current in the PHOKHARA [62] Monte Carlo generator.

Our present results should be regarded as a first step in the study of the  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decays in the framework of  $\text{R}\chi\text{T}$ . In the light of higher statistics for the  $\eta \pi^- \pi^0 \nu_\tau$  mode, or the observation of the decay  $\tau^- \rightarrow \eta' \pi^- \pi^0 \nu_\tau$ , this description could be improved by considering e.g. the exchange of excited vector resonances,  $\text{SU}(3)$  breaking terms in the Lagrangian or revising the  $\eta_8 - \eta_0$  mixing scheme.

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## A Contribution of spin-zero resonances and singlet terms

In this Appendix we analyze both the contribution of spin-zero (scalar and pseudoscalar) resonances and  $\text{SU}(3)$  singlet couplings to  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decays.

### A.1 Scalar and pseudoscalar resonance exchange

Discrete symmetries of QCD constrain the possible couplings in the effective Lagrangian. One of these symmetries is G-parity, which is exact in the  $\text{SU}(2)$  symmetry limit. With our conventions, the corresponding quantum numbers are:  $G_{A_\mu} = -1$ ,  $G_{V_\mu} = +1$ ,  $G_\eta = +1$ ,  $G_{\pi^{(*)}} = -1$ ,  $G_{f_0/\sigma} = +1$ ,  $G_{a_0} = -1$ , thus the final state  $\eta \pi^- \pi^0$  has  $G = +$ . As stated in Sect. 3, since the axial-vector weak current has  $G = -$ , only the vector current can contribute to the  $\tau^- \rightarrow \eta^{(\prime)} \pi^- \pi^0 \nu_\tau$  decay amplitudes in this limit. The intermediate states  $f_0 \pi^-$ ,

$\sigma\pi^-$  are also forbidden by G-parity conservation, which only leaves the channels  $a_0^-\pi^0 \rightarrow \eta\pi^-\pi^0$  and  $a_0^0\pi^- \rightarrow \eta\pi^0\pi^-$ . However, the vector current is  $J^P = 1^-$ , while the intermediate states  $a_0\pi$  have parity  $P = (-1)^{J+1}$ . Therefore, one can conclude that both scalar and pseudoscalar resonance contributions are strongly suppressed at tree level and can be safely neglected.

## A.2 Contribution of double-trace terms

In Ref. [24] two additional operators have been found with respect to those in Ref. [26]. Although these operators involve two traces, hence they are suppressed in the standard counting (in powers of  $p^2 \sim m_q^2$  and  $1/N_C$ ), it is seen that they become leading when a simultaneous counting in all three expansion parameters is carried out [30]. The operators read

$$\begin{aligned}\tilde{O}_{VJP}^8 &= -i\tilde{c}_8 M_V \sqrt{\frac{2}{3}} \varepsilon_{\mu\nu\rho\sigma} \langle V^{\mu\nu} \tilde{f}_+^{\rho\sigma} \rangle \log(\det \tilde{u}) \\ \tilde{O}_{VVP}^5 &= -i\tilde{d}_5 M_V^2 \sqrt{\frac{2}{3}} \varepsilon_{\mu\nu\rho\sigma} \langle V^{\mu\nu} V^{\rho\sigma} \rangle \log(\det \tilde{u}) ,\end{aligned}\quad (\text{A.1})$$

where the tildes stand for  $u$  and  $f$  matrices that include the singlet term (and would contribute to the processes considered here through the  $\eta$  and  $\eta'$  components of the  $\eta_0$  meson). Once again the contribution of these operators vanishes, since the second operator only contributes to neutral current processes, while the first one leads to the contraction of symmetric and antisymmetric tensors in the  $\tau$  decay amplitudes.

## B Isospin relations

In this Appendix we provide a derivation of Eq. (23), which allows to relate the  $\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau$  differential decay rate and the  $\sigma(e^+e^- \rightarrow \eta\pi^+\pi^-)$  cross section.

We work in the limit of SU(2) isospin symmetry, and we neglect  $Z$ -exchange contributions to the hadronic  $e^+e^-$  cross-section, which is a safe approximation in the considered energy range. Thus this process will be driven by the vector current, via photon exchange. One expects to get a relation between this cross section and the vector current contribution to the decay of a tau lepton into the corresponding hadronic state.

Since both  $\eta_8$  and  $\eta_0$  states are SU(2) singlets, we can compute isospin relations between  $\eta_{0,8}\pi\pi$  channels just by taking into account the isospin of  $\pi\pi$  states. Let us denote by  $T_{-0}$ ,  $T_{0-}$  the amplitudes  $\langle \eta\pi\pi | \bar{d}\Gamma_w^\mu u | 0 \rangle$  and by  $T_{+-}$ ,  $T_{-+}$ ,  $T_{00}$  the amplitudes  $\frac{1}{\sqrt{2}} \langle \eta\pi\pi | (\bar{u}\Gamma_w^\mu u - \bar{d}\Gamma_w^\mu d) | 0 \rangle$ , where the subscripts correspond to pion electric charges, and  $\eta$  can be either  $\eta_0$  or  $\eta_8$ . We obtain the relations

$$\begin{aligned}\frac{1}{\sqrt{2}}(T_{-0} + T_{0-}) &= -\frac{1}{\sqrt{6}}(T_{+-} + T_{-+} - 2T_{00}) = 0 , \\ \frac{1}{\sqrt{2}}(T_{0-} - T_{-0}) &= -\frac{1}{\sqrt{2}}(T_{-+} - T_{+-}) , \\ \sqrt{3}(T_{+-} + T_{-+} - T_{00}) &= 0 ,\end{aligned}\quad (\text{B.1})$$

which lead to

$$T_{00} = 0, T_{+-} = -T_{-+} = T_{0-} = -T_{-0}. \quad (\text{B.2})$$

Now let us consider the electromagnetic current. One can decompose it into  $I = 0$  and  $I = 1$  pieces:

$$\Gamma_{em}^\mu = \frac{1}{3} (2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s) = \Gamma_{(0)}^\mu + \Gamma_{(1)}^\mu, \quad (\text{B.3})$$

where

$$\Gamma_{(0)}^\mu = \frac{1}{6} (\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d - 2\bar{s}\gamma^\mu s), \quad \Gamma_{(1)}^\mu = \frac{1}{2} (\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d). \quad (\text{B.4})$$

One can relate the amplitudes  $\langle \eta\pi\pi | \Gamma^\mu | 0 \rangle$  for charge and isospin ( $A_I$ )  $|\eta\pi\pi\rangle$  states by

$$A_{+-} = \frac{1}{\sqrt{2}}A_1 + \frac{1}{\sqrt{3}}A_0, \quad A_{-+} = -\frac{1}{\sqrt{2}}A_1 + \frac{1}{\sqrt{3}}A_0, \quad A_{00} = -\frac{1}{\sqrt{3}}A_0. \quad (\text{B.5})$$

Moreover the vanishing of the amplitude  $A_2$  implies

$$2A_{00} + A_{+-} + A_{-+} = 0. \quad (\text{B.6})$$

In this way one obtains the following relations:

$$\begin{aligned} A_{+-} + A_{00} &= \frac{1}{\sqrt{2}}A_1, \quad A_{-+} + A_{00} = -\frac{1}{\sqrt{2}}A_1, \quad A_1 = \frac{A_{+-} - A_{-+}}{\sqrt{2}}, \\ A_0 &= \frac{A_{+-} + A_{-+} - A_{00}}{\sqrt{3}} = -\sqrt{3}A_{00} = \frac{\sqrt{3}}{2}(A_{+-} + A_{-+}), \end{aligned} \quad (\text{B.7})$$

which lead to

$$|A_{+-} + A_{-+}|^2 + |A_{+-} - A_{-+}|^2 = 2(|A_{+-}|^2 + |A_{-+}|^2) = 4|A_{00}|^2 + 2|A_1|^2, \quad (\text{B.8})$$

$$|A_1|^2 = |A_{+-}|^2 + |A_{-+}|^2 - 2|A_{00}|^2. \quad (\text{B.9})$$

Thus corresponding cross sections are related by

$$\begin{aligned} \sigma(e^+e^- \rightarrow \eta\pi\pi) |_{I=1} &= \sigma(e^+e^- \rightarrow \eta\pi^+\pi^-) + \sigma(e^+e^- \rightarrow \eta\pi^-\pi^+) \\ &\quad - 2 \times 2 \sigma(e^+e^- \rightarrow \eta\pi^0\pi^0) \\ &\simeq 2 \sigma(e^+e^- \rightarrow \eta\pi^+\pi^-), \end{aligned} \quad (\text{B.10})$$

where the additional factor 2 in the first equation arises from the presence of identical particles in the final state. In the last line we have neglected the cross section to the  $\eta\pi^0\pi^0$  state, since it turns out to vanish at the lowest order in the electromagnetic coupling  $\alpha$  owing to  $C$ -parity conservation. For the isoscalar part, from Eq. (B.7) we find

$$\sigma(e^+e^- \rightarrow \eta\pi\pi) |_{I=0} = 6 \sigma(e^+e^- \rightarrow \eta\pi^0\pi^0). \quad (\text{B.11})$$

Finally, one has

$$\frac{1}{\sqrt{2}}(T^{0-} - T^{-0}) = \sqrt{2}T^{0-} = -\langle 1, 0 | \frac{\bar{u}u - \bar{d}d}{\sqrt{2}} | 0 \rangle = -\sqrt{2}A_1. \quad (\text{B.12})$$



Taking into account that the  $e^+e^-$  cross-section into three hadrons is given by

$$\sigma_{e^+e^- \rightarrow h_1 h_2 h_3}(Q^2) = \frac{e^4}{768 \pi^3} \frac{1}{Q^6} \int ds dt |F_3|^2 (-V_{3\mu} V_3^{\mu*}) . \quad (\text{B.13})$$

The cross-sections for the different modes read ( $|A_{+-}|^2 = |A_1|^2/2 = |A_{-0}|^2/2$ )

$$\begin{aligned} \sigma(e^+e^- \rightarrow \eta\pi^+\pi^-) &= \frac{\alpha^2}{96\pi} \frac{1}{Q^6} \int ds dt |T_{-0}|^2 (V_{3\mu} V^{3\mu*}) , \\ \sigma(e^+e^- \rightarrow \eta\pi^0\pi^0) &= \frac{\alpha^2}{48\pi} \frac{1}{Q^6} \int ds dt \frac{1}{2} |T_{00}|^2 (V_{3\mu} V^{3\mu*}) , \end{aligned} \quad (\text{B.14})$$

where the additional factor 1/2 in the second equation arises from the presence of identical particles in the final state. Thus one finally obtains

$$\begin{aligned} \frac{d\Gamma(\tau^- \rightarrow \eta\pi^-\pi^0\nu_\tau)}{dQ^2} &= f(Q^2) \sigma(e^+e^- \rightarrow \eta\pi\pi)|_{I=1} \\ &= 2f(Q^2) [\sigma(e^+e^- \rightarrow \eta\pi^+\pi^-) - 2\sigma(e^+e^- \rightarrow \eta\pi^0\pi^0)] \\ &\simeq 2f(Q^2) \sigma(e^+e^- \rightarrow \eta\pi^+\pi^-) , \end{aligned} \quad (\text{B.15})$$

where  $f(Q^2)$  is the kinematical factor given in Eq. (24).

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